A.L.E. method in MOM6

- ALE is an algorithm
  - ALE enables general coordinate models
- MOM6 interpretation of ALE
  - Eulerian (MOM4/5, MITgcm)
  - ALE from CFD
  - ALE in HyCOM and MOM6
- Other details
  - Pressure gradient must be robust in general coordinates
  - High order reconstructions
- Coordinates in MOM6

[Traditional] Eulerian (fixed grid) algorithm

Ignoring inconvenient details such as barotropic mode, diffusion, etc.

Integrate down from top b.c.

$$\partial_z p = -g\rho(z, S^n, \theta^n) \quad \rightarrow p$$

$$v_h^{n+1} = v_h^n + \Delta t \left(-\frac{1}{\rho_o} \nabla_z p + \cdots \right) \quad \rightarrow v_h^{n+1}$$

$$\partial_z w = -\nabla \cdot v_h^{n+1} \quad \rightarrow w$$

Integrate up from solid bottom

$$\theta^{n+1} = \theta^n - \Delta t \left[\nabla \cdot \left(v_h^{n+1} \theta^n \right) + \partial_z (w\theta^n) + \cdots \right] \rightarrow \theta^{n+1}$$

Explicit vertical transport
Conditionally stable

$$\boxed{\frac{\Delta t w}{\Delta z} < 1}$$

Implications for topographic representation

Arbitrary Lagrangian Eulerian Method (flavor 1)

Applied only in the vertical direction

Hirt et al. , 1974

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

(Leclair & Madec, 2011, use this form)

$$v_h^{n+1} = v_h^n + \Delta t\left(-\frac{1}{\rho_o}\nabla_z p + \cdots\right)$$

$$\delta_k(w^* + w_g) = -\nabla \cdot h^n v_h^{n+1}$$

Specify motion of grid, w_g, here

$$h^{n+1} = h^n + \Delta t \delta_k\,(w_g)$$

Still have explicit transport

$$h^{n+1}\theta^{n+1} = h^n\theta^n - \Delta t\left[\nabla \cdot \left(h^n v_h^{n+1}\theta^n\right) + \delta_k(w^*\theta^n) + \cdots\right]$$

- If $w_g = 0$, then grid is fixed and we recover Eulerian algorithm
- $w^* = w - w_g$ is motion relative to grid
- If we want $w^* = 0$ then we must specify $\partial_z w_g = -\nabla_h . v_h^{n+1}$
- Note that if $w_g \neq 0$ then the grid is moving

Lagrangian phase then remap (flavor 2)

Bleck, 2002

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

$$v_h^\dagger = v_h^n + \Delta t \left(-\frac{1}{\rho_o} \nabla_z p + \cdots \right)$$

$$h^\dagger = h^n - \Delta t \nabla \cdot (h^n v_h^\dagger)$$

Grid moves as if $w^* = 0$

$$h^\dagger \theta^\dagger = h^n \theta^n - \Delta t \left[\nabla \cdot \left(h^n v_h^\dagger \theta^n \right) \right]$$

No vertical transport

- At this point (†), the grid has moved (Lagrangian-ly)
- If the grid is not where we want it, then we remap:

$$h^{n+1} \leftarrow \delta_k Z\left(z^\dagger\right) ; \theta^{n+1} = \theta^\dagger\left(Z(z^\dagger)\right) ; \ldots$$

It's the same thing!

- Specifying Z is potentially simpler than specifying w_g

Eulerian and ALE algorithms side-by-side

Eulerian

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

$$v_h^{n+1} = v_h^n + \Delta t \left(-\tfrac{1}{\rho_o} \nabla_z p + \cdots \right)$$

$$\partial_z w = -\nabla \cdot v_h^{n+1}$$

$$\theta^{n+1} = \theta^n - \Delta t \begin{bmatrix} \nabla \cdot \left(v_h^{n+1} \theta^n \right) + \\ \partial_z (w \theta^n) + \cdots \end{bmatrix}$$

$$\frac{\Delta t w}{\Delta z} < 1$$

A.L.E. (flavor 1)

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

$$v_h^{n+1} = v_h^n + \Delta t \left(-\tfrac{1}{\rho_o} \nabla_z p + \cdots \right)$$

$$\delta_k (w^* + w_g) = -\nabla \cdot h^n v_h^{n+1}$$

$$h^{n+1} = h^n + \Delta t \delta_k (w_g)$$

$$h^{n+1} \theta^{n+1} = h^n \theta^n - \Delta t \begin{bmatrix} \nabla \cdot \left(h^n v_h^{n+1} \theta^n \right) \\ + \delta_k (w^* \theta^n) + \cdots \end{bmatrix}$$

$$\frac{\Delta t w^*}{\Delta z} < 1$$

$$w^* = w - w_g$$

A.L.E. (flavor 2)

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

$$v_h^{\dagger} = v_h^n + \Delta t \left(-\tfrac{1}{\rho_o} \nabla_z p + \cdots \right)$$

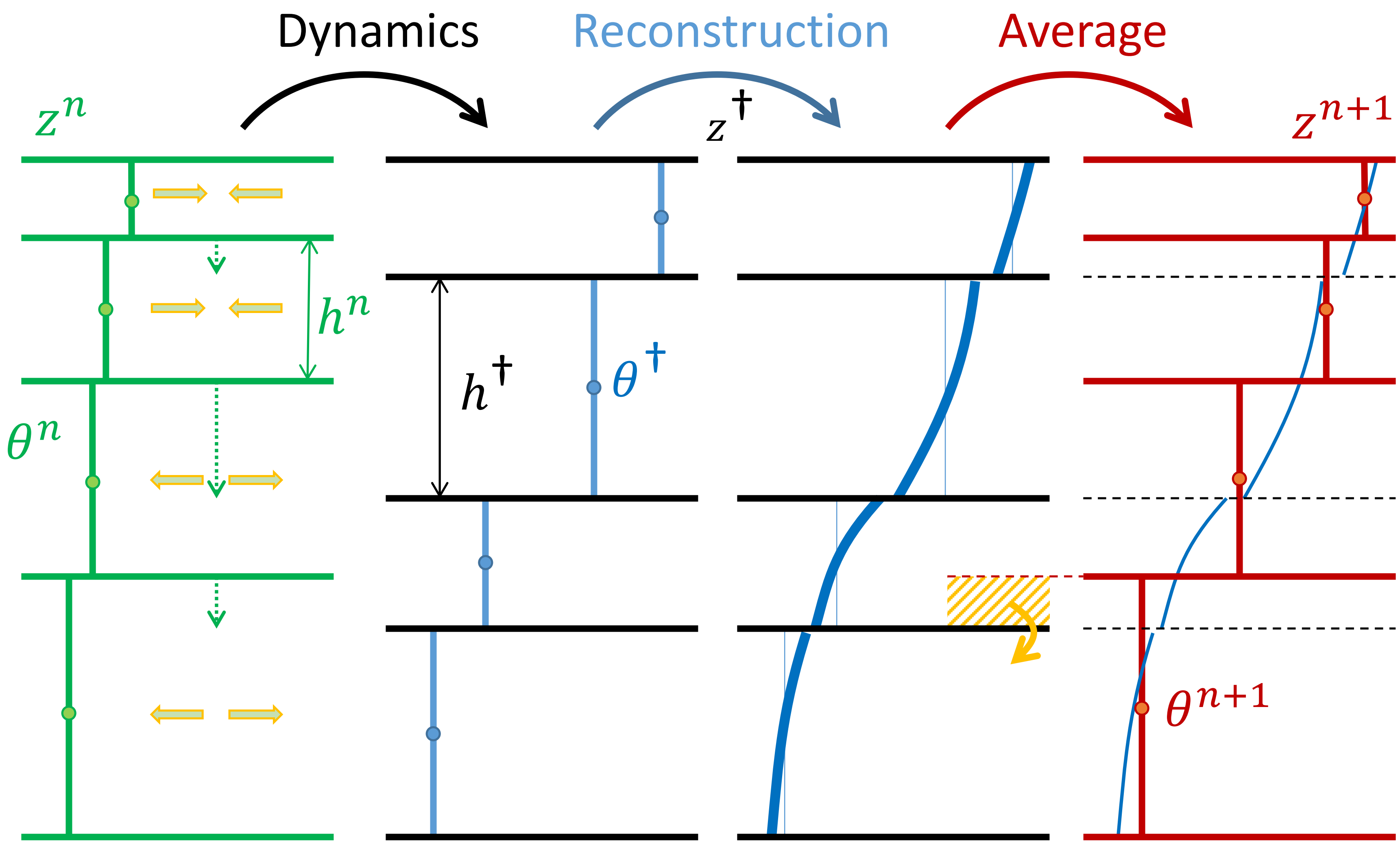
$$h^{\dagger} = h^n - \Delta t \nabla \cdot (h^n v_h^{\dagger})$$

$$h^{\dagger} \theta^{\dagger} = h^n \theta^n - \Delta t \left[\nabla \cdot \begin{pmatrix} h^n v_h^{\dagger} \theta^n \\ + \cdots \end{pmatrix} \right]$$

$$h^{n+1} \leftarrow \delta_k Z \left(z^{\dagger} \right)$$

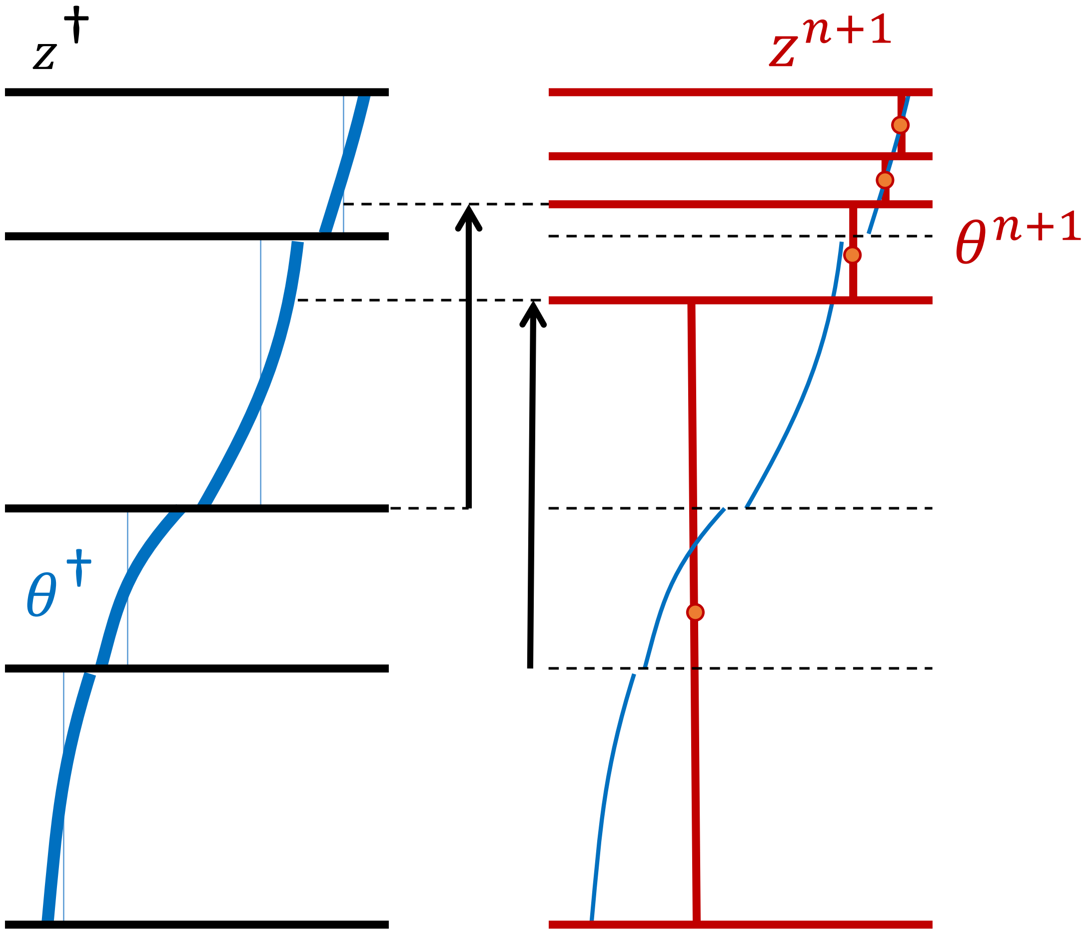
$$\theta^{n+1} = \theta^{\dagger} \left(Z(z^{\dagger}) \right)$$

Full A.L.E. (flavor 2) time step



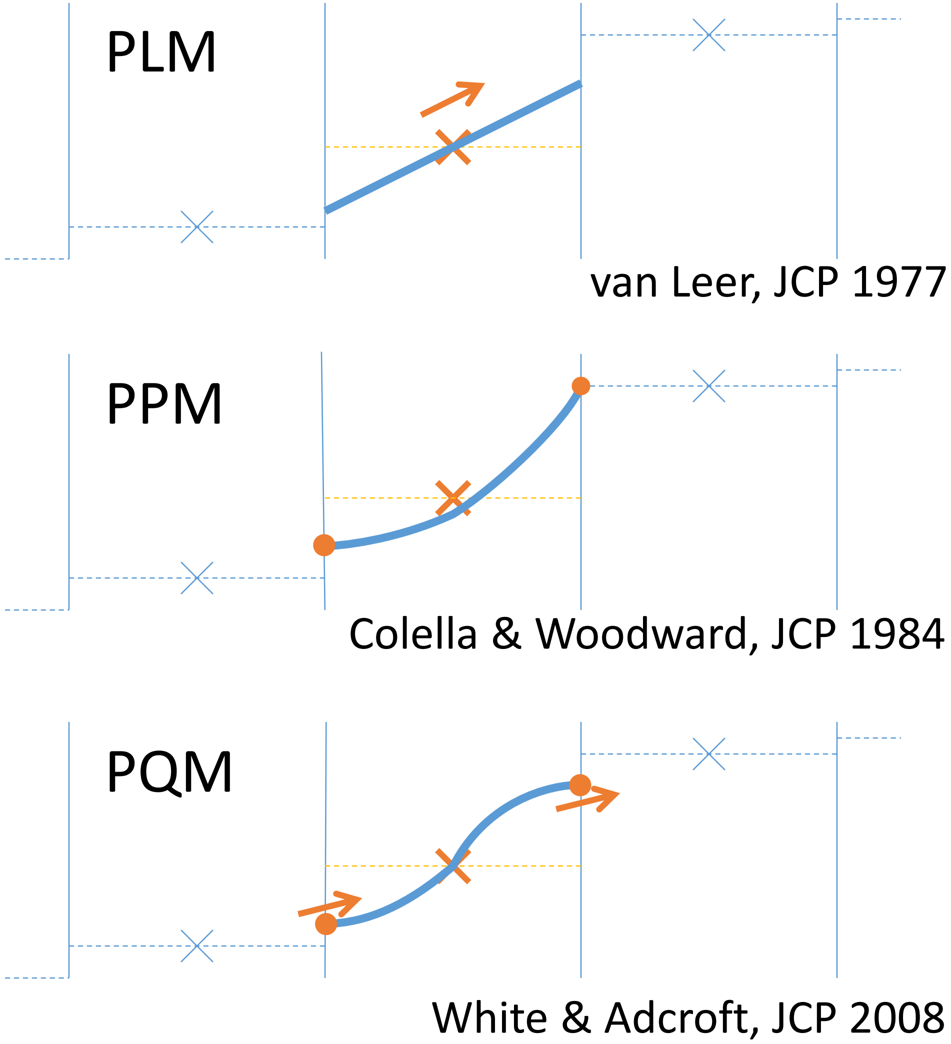
Stable to “CFL>1” (vertical)

- Target grid, z^{n+1}, can be ANY grid
  - can even have different # of levels
- No restrictions on $\Delta t w_g$ or $\Delta t w$
- Remapping by “projection” works but is prone to non-conservation (roundoff)
- Casting remapping in flux form connects back to flavor 1
  - Remapping equivalent to advection
- Accuracy determined by choice of reconstruction



Piecewise * Method (* = C,L,P or Q)

- PLM: two degrees of freedom
  - Cell mean + slope
- PPM: three degrees of freedom
  - Very widely used
  - Cell mean + two edge values
- PQM: five degrees of freedom
  - Cell mean + two edge values + two edge slopes



Successive schemes provide more flexibility to represent structures → more accurate

Sub-cycling dynamics

$$\partial_z p = -g\rho(z, S^n, \theta^n)$$

$$v_h^* = v_h^n + \Delta t \left(-\frac{1}{\rho_o}\nabla_z p + \cdots\right)$$

$$h^* = h^n - \Delta t \nabla \cdot (h^n v_h^*)$$

Internal gravity waves

$$\frac{\Delta t c_g}{\Delta x} < 1$$

$\times\, M$

$$h^*\theta^* = h^n\theta^n - M\Delta t \left[\nabla \cdot \left(\sum_{m=1}^{M} h^n v_h^* \theta^n\right)\right]$$

$$\frac{M\Delta t u_h}{\Delta x} < 1$$

$\times\, N$

$$h^{n+1} \leftarrow \delta_k Z(z^*)\,;\, \theta^{n+1} = \theta^*(Z(z^*))\,;\, \ldots$$

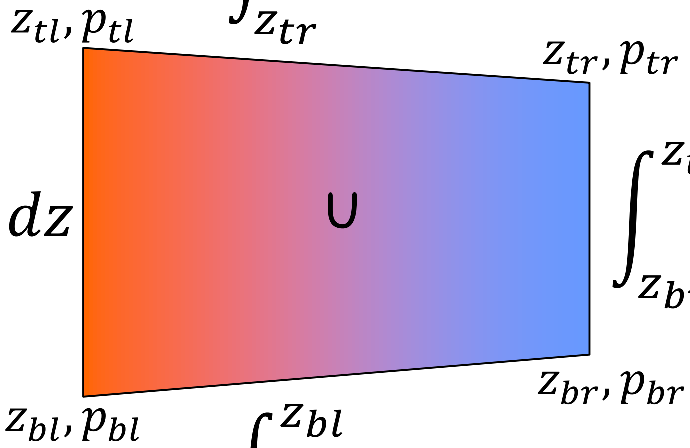
From Robert Hallberg, GFDL

PGF error

- "Analytically" integrate FV PGF
  - Necessary in isopycnal ocean model to avoid thermobaric instability
  - Numerical quadrature more practical

$$\int_{z_{tr}}^{z_{tl}} p\, dz$$

$$\int_{z_{bl}}^{z_{tl}} p\, dz$$

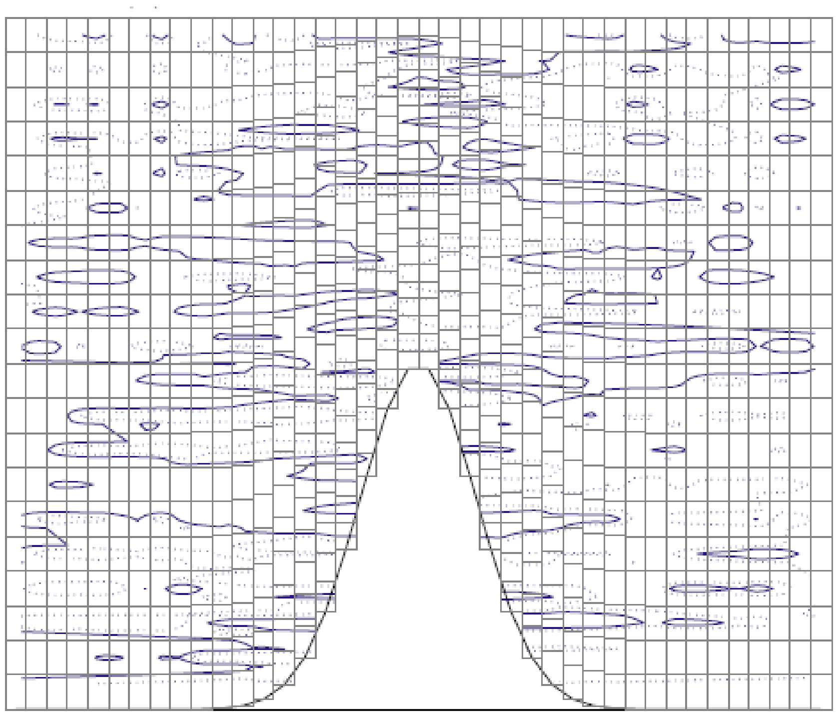


$$\int_{z_{br}}^{z_{tr}} p\, dz$$

$$\int_{z_{br}}^{z_{bl}} p\, dz$$

$$p_{bl} = p_{tl} + \int_{z_{bl}}^{z_{tl}} g\rho\, dz \qquad p_{br} = p_{tr} + \int_{z_{br}}^{z_{tr}} g\rho\, dz$$

Seamount resting ocean test



C.I. = 10^{-13} m/s, max |u| ~ 10^{-11} m/s

Adcroft et al., 2008; White et al. 2009

Benefits of A.L.E.

- Can explore alternative-/general-/hybrid-coordinates
  - Long running debate about “best” coordinate

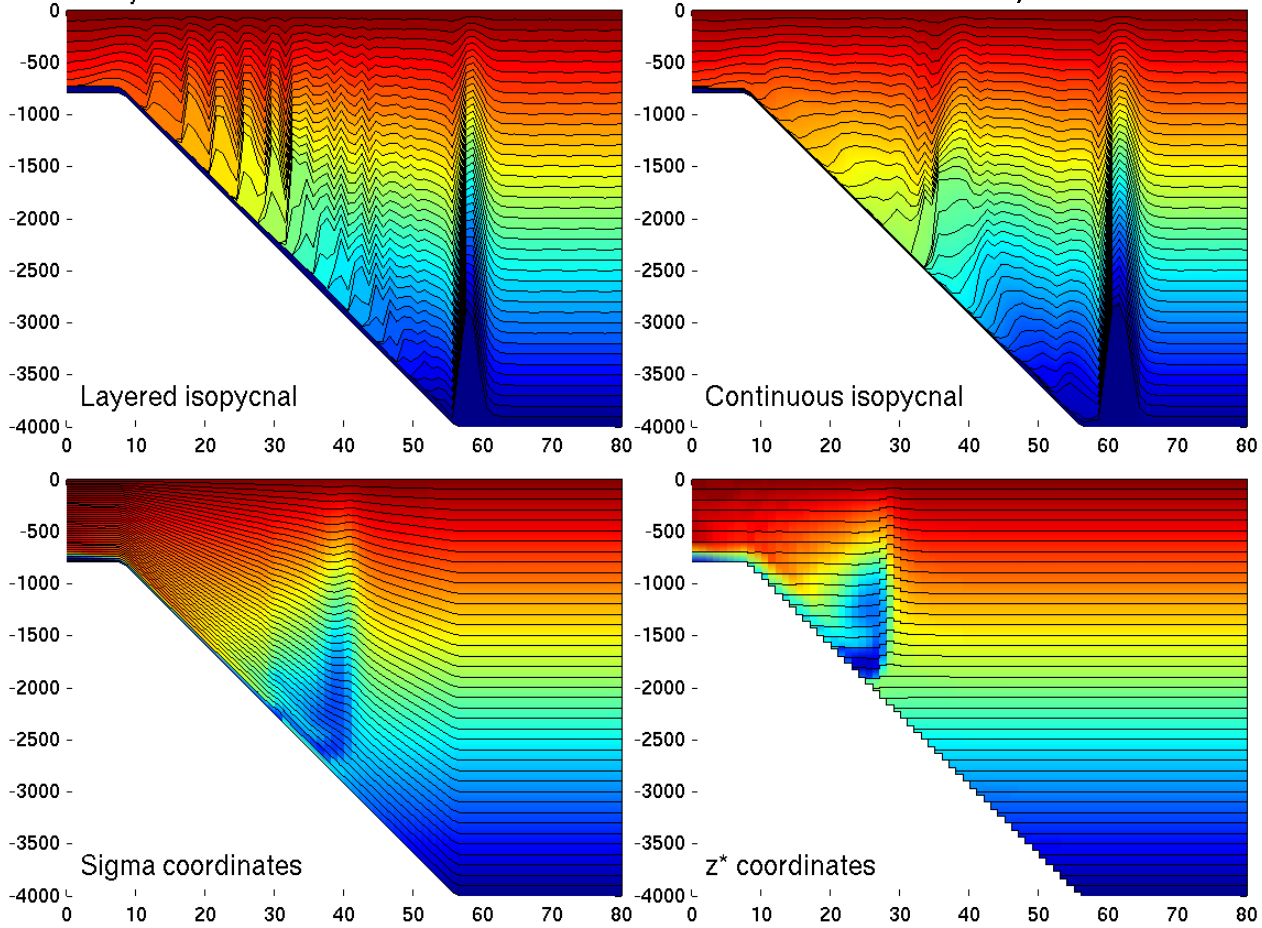
    (ignore the debate and the question)
  - Adaptive/flexible resolution very useful
- Sub-cycling can offer significant efficiencies
- Accurate and robust to vertical motion
- No need to compromise topography or resolution (unconditionally stable)

- Writing code for general coordinates requires extra thought
  - e.g. parameteriztions might be specific to coordinate

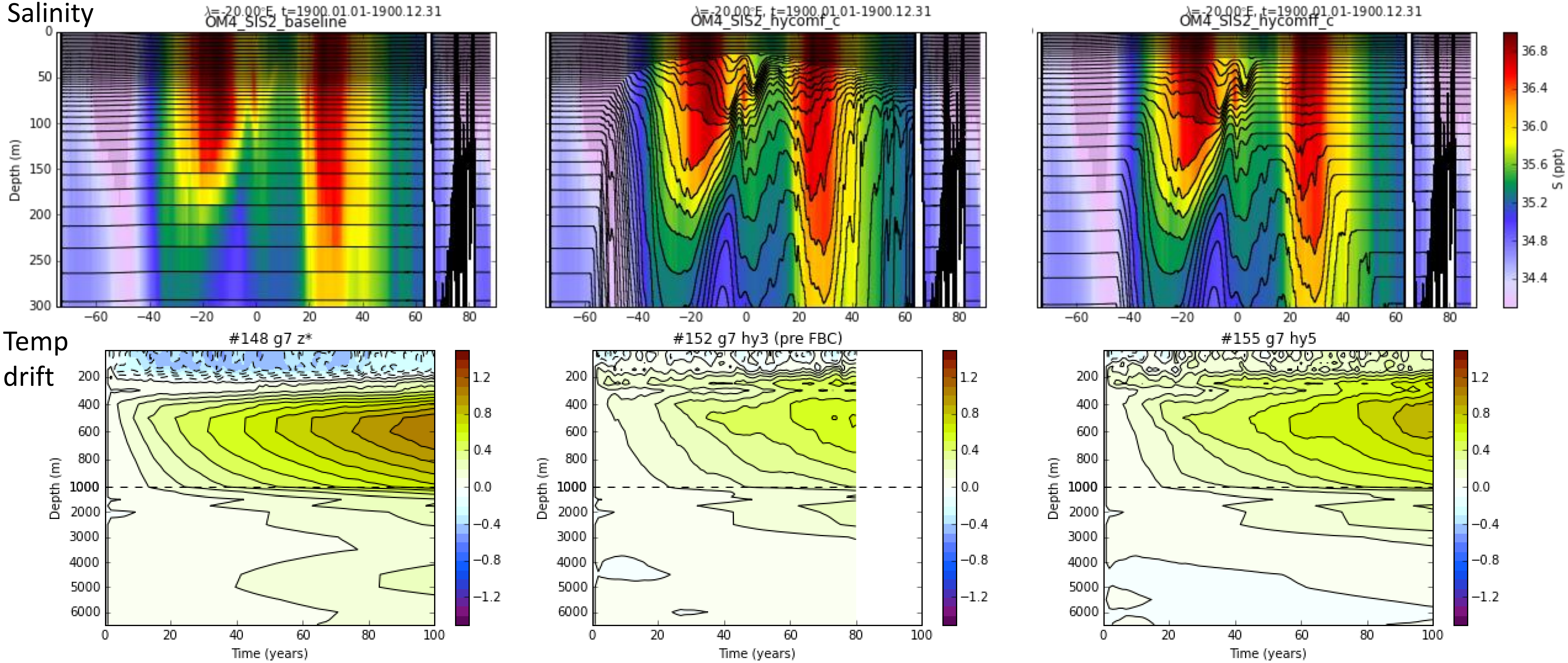
MOM6 coordinates

- Layered isopycnal
  - traditional, not ALE
- Continuous isopycnal (ρ_2)
- Geopotential (z*)
- Terrain following (σ)
- Hybrid “HYCOM1”
  - Deeper of z* and ρ_2 for each k
- “SLIGHT”
  - Attempt to use less z* space than HYCOM1 (not yet successful)
- More to be coded…

White et al., JCP 2009



Climate drift as function of ocean coordinate



Chassignet et al., 2003; Megann et al., 2010; Ilicak et al., 2012